(25) $\dfrac{n(3n+1)}{4(n+1)(n+2)}$. (26) $\dfrac{n(n+1)}{4(n+2)(n+3)}$.

(27) $\dfrac{n(n+2)}{3(2n+1)(2n+3)}$. (28) $\dfrac{n(n+1)}{4(3n+2)(3n+5)}$.

(29) $\dfrac{1}{3}\left[\dfrac{1}{3!}-\dfrac{1}{(n+1)(n+2)(n+3)}\right]$. (30) $\dfrac{n^2(n+1)^2(2n^2+2n-1)}{3.4}$.

(31) $\dfrac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{6.7}$.

(32) $(n-1)2^n+1$. (33) $\dfrac{(2n-1)3^n+1}{4}$.

(34) $\dfrac{(4n-1)5^n+1}{16}$. (35) $(2n-3)2^n+3$.

(36) $\dfrac{(6n-7)3^n+7}{4}$. (37) $\dfrac{(12n-13)4^n+13}{9}$.

(38) $(n^2-2n+3)2^n-3$. (39) $\dfrac{(n^2-n+1)3^n-1}{2}$.

(40) $[(n-1)^3-6(n-2)]2^n+13$. (41) $2^{n-1}(n^2-n+2)-1$.

(42) $\dfrac{3^{n+1}-(n^2+3n+3)}{2.3^{n-1}}$. (43) $\dfrac{n(n+1)(3a+2n-2)}{6}$.

(61) $\dfrac{\sin\dfrac{na}{2}\sin\dfrac{(n+1)a}{2}}{\sin\dfrac{a}{2}}$. (62) $\dfrac{\sin^2 na}{\sin a}$.

(63) $\dfrac{\sin\dfrac{na}{2}\cos\dfrac{(n+1)a}{2}}{\sin\dfrac{a}{2}}$. (64) $\dfrac{\sin 2na}{2\sin a}$.

ON THE GENERAL TANGENT TO PLANE CURVES.*

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The object of this note is to work out without the Calculus a certain well known formula for a tangent to a plane algebraic curve at an ordinary (not a singular) point, and especially to show how this result is easily extended to the loci of transcendental equations. The formula found is readily developed by aid of the Differential Calculus, but is here found by other means. It might therefore be used in a course which does not presuppose the Calculus.

Let us take, in rectangular Cartesian coordinates, the general equation of a proper nth degree locus in the form

*Read before the Chicago Section of the American Mathematical Society, April 18, 1908.

$$[1] \qquad a_{00}+a_{10}x+a_{01}y+a_{20}x^2+a_{11}xy+a_{02}y^2+\ldots+\sum_{s=0}^{s=n} a_{n-s,\,s}\,x^{n-s}y^s=0.$$

Let [α, β] be an ordinary point of the curve, so that at this point only one tangent to the curve can be drawn. Transform the equation by the substitution

$$[2] \qquad x=x'+\alpha, \quad y=y'+\beta.$$

The new origin will be on the curve; and therefore the independent term vanishes. The resulting equation takes the form

$$[3] \qquad a_{10}x'+a_{01}y'+2a_{20}\alpha\, x'+a_{11}\beta\, x'+a_{11}\alpha\, y'+2a_{02}\beta\, y'+\ldots$$
$$+\sum_{s=0}^{s=n} a_{n-s,\,s}\left\{[n-s]\alpha^{n-s-1}\beta^s x'+s\,\alpha^{n-s}\beta^{s-1}y'\right\}$$
$$+\text{terms of second degree and higher in } x',\ y'=0.$$

Let $u^{(p)}$ be identically equal to the sum of all the terms of degree p in x', y' in this equation. Then the equation takes the form

$$[3_a] \qquad u^{(1)}+u^{(2)}+u^{(3)}+\ldots+u^{(n)}=0.$$

Then is $u^{(1)}=0$ the equation of the tangent to this curve at the origin. (See Salmon-Fiedler, *Hoeheren ebenen Kurven*, 2nd ed., p. 31.) For since the origin is now an ordinary point of the curve, $u^{(1)}$ is not identically equal to zero. [If the origin were a double point $u^{(1)}$ would be identically equal to zero. Then in order to find the equations of the tangents it would be necessary to deal with $u^{(\sigma)}$, where $u^{(\sigma)}$ is the first of the quantities $u^{(2)}$, $u^{(3)}$, ... which is not identically zero. See Salmon-Fiedler, *l. c.* This case is excluded from the present discussion by the first assumption that (α, β) is an ordinary point of the locus of [1]].

Now from [3] it is easy to write

$$u^{(1)}\equiv\sum_{n=1}^{n=n}\sum_{s=0}^{s=n} a_{n-s,\,s}\left\{[n-s]\alpha^{n-s-1}\beta^s x'+s\,\alpha^{n-s}\beta^{s-1}y'\right\}=0;$$

or,

$$\frac{y'}{x'}=-\frac{\displaystyle\sum_{n=1}^{n=n}\sum_{s=0}^{s=n} a_{n-s,\,s}[n-s]\alpha^{n-s-1}\beta^s}{\displaystyle\sum_{n=1}^{n=n}\sum_{s=0}^{s=n} a_{n-s,\,s}\,s\,\alpha^{n-s}\beta^{s-1}},$$

as the equation of the tangent to the locus of [3] at the origin. Hence by [2], the tangent to the locus of [1] at the point [α, β] is

$$[4] \qquad \frac{y-\beta}{x-\alpha} = -\frac{\sum\limits_{n=1}^{n=n}\sum\limits_{s=0}^{s=n} a_{n-s,\,s}\,[n-s]\,\alpha^{n-s-1}\beta^{s}}{\sum\limits_{n=1}^{n=n}\sum\limits_{s=0}^{s=n} a_{n-s,\,s}\,s\,\alpha^{n-s}\beta^{s-1}}.$$

EXAMPLE 1. Applying this to the circle $x^2+y^2=r^2$, we have for its tangent at $[\alpha, \beta]$

$$\frac{y-\beta}{x-\alpha} = -\frac{2\alpha}{2\beta} = -\frac{\alpha}{\beta}.$$

Since $\alpha^2+\beta^2=r^2$ this reduces to the usual formula

$$x\alpha+y\beta=r^2.$$

EXAMPLE 2. In the conic $a_{20}x^2+a_{11}xy+a_{02}y^2+a_{10}x+a_{01}y+a_{00}=0$, we have

$$\frac{y-\beta}{x-\alpha} = -\frac{2a_{20}\alpha+a_{11}\beta+a_{10}}{2a_{02}\beta+a_{11}\alpha+a_{01}}$$

for the tangent at $[\alpha, \beta]$. Since $[\alpha, \beta]$ is on the locus, it is easy to reduce this equation to the form

$$[2a_{20}\alpha+a_{11}\beta+a_{10}]x+[2a_{02}\beta+a_{11}\alpha+a_{01}]y+a_{10}\alpha+a_{01}\beta+2a_{00}=0,$$

the general equation of a tangent to a conic.

Now, in the equation

$$y-\sin x\,\cos y=0$$

suppose that $\sin x$ and $\cos y$ are replaced by their expansions in series and that the product is taken. The resulting equation with infinite number of terms, may evidently be dealt with by the same method as that which we have used with reference to [1]. And so of the locus of any such equation. Instead of formula [4] it is evident that we should in these cases have the tangent at $[\alpha, \beta]$ determined by the following equation:

$$[5] \qquad \frac{y-\beta}{x-\alpha} = -\frac{\sum\limits_{n=1}^{n=\infty}\sum\limits_{s=0}^{s=n} a_{n-s,\,s}\,[n-s]\,\alpha^{n-s-1}\beta^{s}}{\sum\limits_{n=1}^{n=\infty}\sum\limits_{s=0}^{s=n} a_{n-s,\,s}\,s\,\alpha^{n-s}\beta^{s-1}},$$

provided both infinite series are convergent.

If the equation of the given locus is $y-f(x)=0$ where f denotes an algebraic function of x or an infinite convergent series of terms containing x only in positive integral powers and if the coefficients are represented by the same quantities as before; then equations [4] and [5] become

$$[6] \qquad \frac{y-\beta}{x-\alpha} = -\sum_{n=1}^{n=n} a_{n,\,0}\, n\, \alpha^{n-1}$$

where the range of n is finite or infinite according as $f(x)$ is or is not an algebraic function of x. In the latter case, of course, the series of [6] must be convergent for the given value of α in order that the formula may be employed.

It may be pointed out that formulae [4], [5], [6] can be obtained by differentiation and the substitution of α, β in the results. Thus if $u=0$ is the equation of the curve, these become, respectively,

$$\frac{y-\beta}{x-\alpha} = -\frac{D_x u}{D_y u}; \quad \frac{y-\beta}{x-\alpha} = -\frac{D_x u}{D_y u}; \quad \frac{y-\beta}{x-\alpha} = -D_x u,$$

where x and y are replaced by α and β in every $D_x u$ and $D_y u$. Hence, the interest which attaches to the discussion in this note is not in the results themselves but in the method of obtaining them without the aid of the Calculus.

EXAMPLE 3. What is the tangent to the curve $y=\sin x$ at the point $[\alpha, \beta]$?

We suppose that this problem is assigned to a pupil who is yet unacquainted with the Calculus, but one who knows from Trigonometry the series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots.$$

The given equation becomes

$$y = x - \frac{x^2}{3!} + \frac{x^5}{5!} - \ldots,$$

and therefore by [6] the equation to the tangent is

$$\frac{y-\beta}{x-\alpha} = 1 - \frac{\alpha^2}{2\,!} + \frac{\alpha^4}{4\,!} - \ldots$$

But the series of the second member is equal to $\cos \alpha$. Hence for the required tangent we have finally the equation

$$y=\beta+[x-\alpha]\cos\alpha.$$

EXAMPLE 4. Similarly, the tangent to $y=\cos\alpha$ at the point $[\alpha, \beta]$ is

$$y=\beta-[x-\alpha]\sin\alpha.$$

EXAMPLE 5. What is the tangent to $y=\tan x$ at the point $[\alpha, \beta]$?

EXAMPLE 6, What is the tangent to $y=\log_e[1+x]$ at the point $[\alpha, \beta]$? We have

$$y=\log_e[1+x]=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots, \quad |x|<1;$$

and therefore [6] yields

$$\frac{y-\beta}{x-\alpha}=1-\alpha+\alpha^2-\alpha^3+\ldots=\frac{1}{1+\alpha}, \text{ since } |\alpha|<1.$$

Thus, for the required equation we have

$$y=[1+\alpha]x+\beta-\alpha^2.$$

EXAMPLE 7. Similarly, the tangent to $y=\log_e[1-x]$ at the point $[\alpha, \beta]$ is

$$y=[1-\alpha]x+\beta+\alpha^2.$$

[Remark. The chief difficulty in using the foregoing method with pupils beginning the study of Analytics will be in the discussion following equation $[3_\alpha]$ above. A reference to Salmon (or Salmon-Fiedler, *l. c.*) will suffice to show that the difficulty is by no means insurmountable. If the tangent is defined as a straight line which passes through two consecutive points, the method by which Salmon shows that our $u^{(1)}=0$ is the tangent to our $u^{(1)}+u^{(2)}+\ldots+u^{(n)}=0$ will be easily within the grasp of the earnest student. Nothing else in the paper presents any intrinsic difficulty whatever. And the bright student will undoubtedly take interest in a simple general method and formula for solving numerous special problems which are continually arising.]